Journal of Statistical Physics, Vol. 58, Nos. 3/4, 1990

High-Order Cycles in the Logistic Map or Centers of Cardioids in the Mandelbrot Set

John Stephenson[1]

Received July 14, 1989

Asymptotic expressions for the positions of the centres (along the real axis) of high-order cycle cardioids closest to the limit point -2 in the Mandelbrot set are obtained by a combination of numerical and analytical methods.

KEY WORDS: Logistic map; Mandelbrot set.

1. INTRODUCTION

In two important review articles Domb[1] has provided a survey of series expansion methods in critical phenomena, and has shown how the asymptotic behavior of the (often integer) coefficients comprising a series could yield information about the location of a critical point and the values of the exponents and amplitudes associated with continuous (second- and higher-order) transitions. When the coefficients varied smoothly, examination of the ratios of coefficients (the "ratio method") was especially easy to use, and enabled Domb and Sykes[2] to make some of the first precise estimates of the exponents of the susceptibility of the three-dimensional Ising model.

In this paper I examine a problem which arose while studying the Mandelbrot set,[3] which contains a sequence of cardioids along the real axis. The positions of the centres of the "principal" cardioids form an interesting monotonic sequence which can be analyzed by the ratio method. These cardioids occur in the "periodic windows" of the (real) logistic map. By using the ratio method, one can obtain an expression for the asymptotic positions of the centres of the cardioids, which can be confirmed (partly analytically and partly numerically) from Feigenbaum's form of the logistic

[1] Physics Department, University of Alberta, Edmonton, Alberta, Canada, T6G 2J1.

0022-4715/90/0200-0579$06.00/0

map. Various exponents and amplitudes are estimated. Finally, an exact solution of the principal cardioid problem is presented.

A particularly fine introduction to the Mandelbrot set and many related aspects of the logistic map has been given by Peitgen and Richter[4] in a book where references to many original papers will be found. I assume the reader is familiar with the Mandelbrot set and the associated logistic map through this work, or some other of the many excellent texts now available. The section headings indicate the layout of this paper.

2. PRINCIPAL CARDIOIDS IN THE MANDELBROT SET

The quadratic iterative map

$$z_{n+1} = R_c(z_n) = z_n^2 + c, \qquad n = 0, 1, 2, \ldots, \quad c \text{ complex} \tag{1}$$

yields the Mandelbrot set as the set of complex c for which iterates starting at the origin ($z_0 = 0$) remain bounded. After n iterations one has (dropping the subscript c)

$$z_n = R(z_{n-1}) = R(R(R(\cdots R(z_0)))) \equiv R^n(z_0) \tag{2}$$

For real values of the parameter c, iterative models of the form (1) exhibit bifurcation leading to the onset of chaos.

The quadratic map is equivalent to the logistic map, which (for real variables) May[5] and Feigenbaum[6] have written in the form

$$x_{n+1} = ax_n(1 - x_n), \qquad a \text{ real} \tag{3}$$

The algebraic relation between these forms is

$$(a-1)^2 = 1 - 4c, \qquad z_n = a(\tfrac{1}{2} - x_n) \tag{4}$$

Those parts of the Mandelbrot set originating in cycles of finite order n, so

$$z_0 = z_n = R^n(z_0), \tag{5}$$

form smooth figures, such as cardioids and bubbles. $n = 1$ gives the main cardioid, $n = 2$ gives a circle attached to the main cardioid, and the cycles $n = 3$ and 4 each give rise to a cardioid on the real axis. For $n = 5$ there are three cardioids on the real axis, and so on. Each cardioid on the real axis locates a periodic window where a new cycle of order n commences. To the left of each cardioid a sequence of bubbles corresponding to (real axis) bifurcations occurs, as described by Feigenbaum.[6] The higher-order cardioids all lie between the Myrberg point $-1.4011552\ldots$, where the bifurcation sequence associated with the main cardioid terminates, and the "spike" of the Mandelbrot set at the limit point -2. For each cycle of order n the cardioid closest to -2 will be called the "principal cardioid."

It is straightforward to calculate all the cardioids exactly (algebraically) for low orders of cycles (e.g., $n = 1, 3, 4, 5$). One observes that as n increases, the principal cardioids are all similar in shape and shrink in size while approaching the limit point -2. An eyeball estimate suggests that the distance of each principal cardioid from -2 decreases by a scaling factor of 4 at each stage.

As a first step toward determining the scaling factor and the nature of the approach of the principal cardioids to the limit point, we study the positions of the "centres" of the cardioids, which are located[4] by the values of c for which the origin is a member of the cycle in question. The required values of c at the centres can be located as the zeros of polynomials

$$R_c^n(0) \equiv R_c^{n-1}(c) = 0 \tag{6}$$

which can easily be constructed by direct algebraic iteration. For example

$$R_c^2(0) = c^2 + c, \qquad R_c^3(0) = c^4 + 2c^3 + c^2 + c \tag{7}$$

Since the size of coefficients and degree (2^{n-1}) of each polynomial increase rapidly with n, this method is only feasible for low-order cycles. So in

Table I. Values of c at Centers of Principal Cardioids Corresponding to cycles of Order n[a]

n	c_n				
3	−1.7548776	6624669	2760049	5088963	5852870
4	−1.9407998	0652948	4752232	0909796	5520419
5	−1.9854242	5305420	5310609	7505827	1867435
6	−1.9963761	3771119	3750644	8798190	6060663
7	−1.9990956	8232701	8473210	6299992	2229446
8	−1.9997740	4869372	7323471	7009961	3091576
9	−1.9999435	2176567	4009146	1790814	9080053
10	−1.9999858	8114039	2107911	5315548	1791579
11	−1.9999964	7033500	8689607	2273726	5654157
12	−1.9999991	1758726	0825033	3507285	8064714
13	−1.9999997	7939705	8828114	4878867	8768003
14	−1.9999999	4484928	1454161	7619352	9701129
15	−1.9999999	8621232	1505285	6666396	6496980
16	−1.9999999	9655308	0453621	1087401	5473483
17	−1.9999999	9913827	0118607	7969200	4616478
18	−1.9999999	9978456	7530000	3122841	4657128
19	−1.9999999	9994614	1882523	3011102	5999467
20	−1.9999999	9998653	5470632	3673642	9077819
21	−1.9999999	9999663	3867658	1938869	1650548
22	−1.9999999	9999915	8466914	5552036	8333112

[a] For $n > 7$ up to 42 decimal places were used.

practice one computes the centres numerically to high precision by direct iteration. The first 20 principal centres for $n = 3$–22, given in Table I, form a smooth sequence, which I analyze by the ratio method, as follows.

3. THE RATIO METHOD

By the usual hypothesis (as in critical phenomena), I assume that to leading order the asymptotic form of the terms in a sequence a_n is

$$a_n \sim a_\infty + A/G^n n^\varepsilon \tag{8}$$

where a_∞, A, G, and ε are the critical "value," "amplitude," "constant," and "exponent," respectively. Without prejudice to the values of a_∞ and A, the constant G and the exponent ε can be extracted by plotting (following Domb,[1] Domb and Sykes,[2] and also Feigenbaum[6])

$$G_n = (a_n - a_{n+1})/(a_{n+1} - a_{n+2}) \sim G(1 + \varepsilon/n) \tag{9}$$

versus $1/n$, so G is the intercept and $G\varepsilon$ the slope of the resulting "$1/n$ plot."

For the centres of the principal cardioids we calculate the differences, ratios of differences, and (following Feigenbaum[6]) the ratios of differences of ratios:

$$d_n = c_n + 2, \qquad r_n = d_n/d_{n+1}, \qquad s_n = (r_n - r_{n+1})/(r_{n+1} - r_{n+2}) \tag{10}$$

It is clear numerically that $c_n \to -2$, $d_n \to 0$, and $r_n \to 4$ exponentially fast. In the present case it is sometimes more convenient to number the terms by $m = n - 2$, so if we write $c_n \sim c_\infty + A/G_0^m m^{\varepsilon_0}$, so

$$r_n = d_n/d_{n+1} \sim G_0(1 + \varepsilon_0/n) \tag{11}$$

then $c_\infty = -2$, $G_0 \equiv r_\infty = 4$, $\varepsilon_0 = 0$, regardless of A.

Consequently, the first genuine "$1/n$ plot" comes from

$$r_n \sim r_\infty + B/G_1^m m^{\varepsilon_1} \qquad \text{with} \qquad s_n \sim G_1(1 + \varepsilon_1/n), \qquad m = n - 2 \tag{12}$$

in Fig. 1, where the intercept is $G_1 \sim 4$, and, from the slope, $\varepsilon_1 \sim -1$. [In Fig. 1, s_n is plotted against $1/(n-1)$, $1/n$, $1/(n+1)$ in the upper, middle, and lower graphs, respectively.] It is also "obvious" numerically that $r_\infty = 4$, and in fact a "$1/n$ plot" of $[(r_n - 4)/(r_{n+1} - 4)]$ is very similar to Fig. 1.

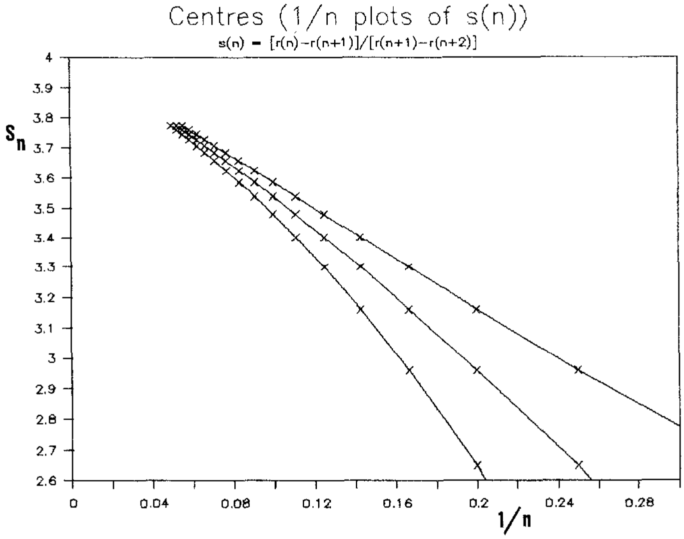


Fig. 1. A $1/n$ plot of s_n, defined in (10), against $1/(n-1)$, $1/n$, and $1/(n+1)$ in the upper, middle, and lower plots, respectively. The intercept is $G_1 \sim 4$, and, from the slope, $\varepsilon_1 \simeq -1$.

4. ESTIMATION OF AMPLITUDES

The determination of the amplitudes is a messy process (below), and the final results are in (23) and (24). Assuming now that $G_0 = G_1 = r_\infty = 4$, we estimate the amplitudes A and B by calculating

$$A_n = 4^{n-2} d_n, \text{ and } B_n, \text{ defined in (15) below} \tag{13}$$

as in Table II. The approach of A_n as a decreasing sequence to a limiting value close to $A = 0.925275412602$ is exponentially fast, as may be confirmed by a graph of A_n versus $m/4^m$ [in which the choice of abscissa is based on the asymptotic behavior of C_n below in (20)], and by the behavior of r_n.

The estimation of B could start by writing r_n in (12) as

$$r_n = r_\infty + B_n / G_1^m m^{\varepsilon_1}, \qquad \text{where} \quad B_n \to B \text{ as } n \to \infty \tag{14}$$

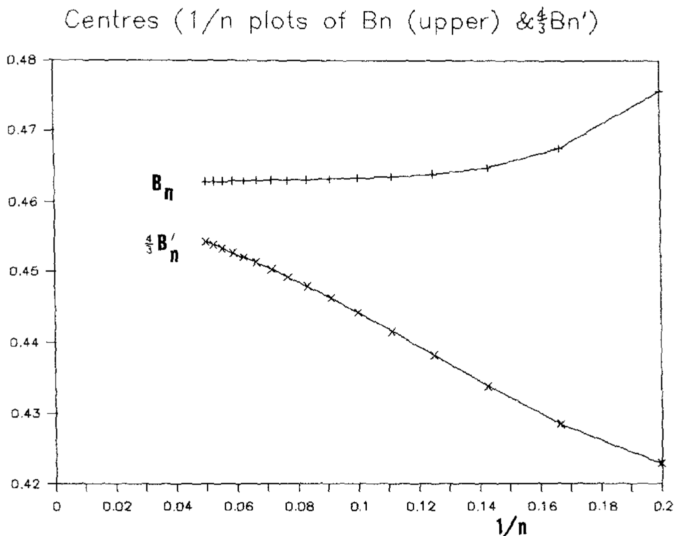
and calculating the combination (defining B'_n)

$$G_1^m (r_n - r_{n+1})\, m^{\varepsilon_1} = B'_n = B_n [1 - 1/G_1 (1 + 1/m)^{\varepsilon_1}] \tag{15}$$

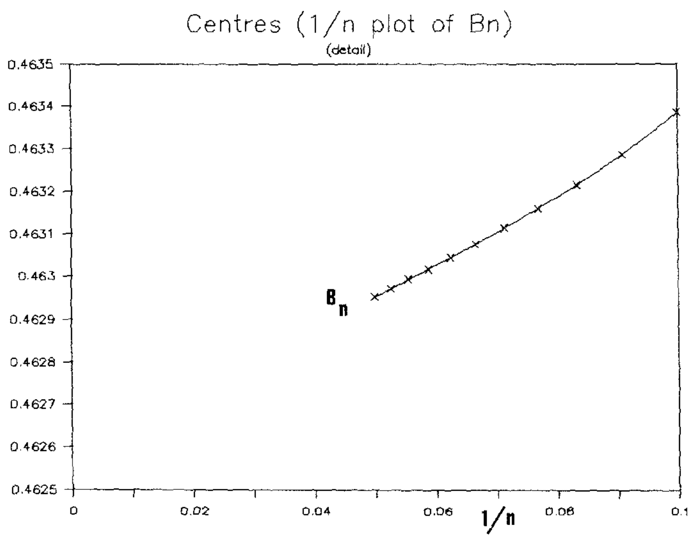
assuming $\varepsilon_1 = -1$ and $G_1 = 4$. This yields the lower graph of $\frac{4}{3} B'_n$ [$= B'_n/(1 - 1/G_1)$, with the awkward unavoidable factor $1/(1 - 1/G_1) = \frac{4}{3}$

Table II. Sequences for Coefficients in the Asymptotic Forms for c_n in (23) and (24)

n	A_n	$(4/3)\ B'_n$	B_n	$(4/3)\ C'_n$	C_n	$4^m(A_n - A_{n+1})$
3	0.98048933501323	0.4213959	0.6320939	0.1775266	0.2662899	0.133145
4	0.94720309552824	0.4202307	0.5042768	0.1531231	0.1837477	0.229685
5	0.93284780453086	0.4229416	0.4758093	0.1461777	0.1644499	0.328900
6	0.92770874593440	0.4284647	0.4674161	0.1439956	0.1570861	0.431987
—	—	—	—	—	—	
17	0.92527541464137	0.4527283	0.4630175	0.1428036	0.1460492	1.606541
18	0.92527541314516	0.4533476	0.4629933	0.1427965	0.1458347	1.713558
19	0.92527541274619	0.4538941	0.4629720	0.1427902	0.1456460	1.820575
20	0.92527541264022	0.4543798	0.4629530	0.1427846	0.1454786	1.927591
21	0.92527541261217	—	—	0.1427795	0.1453292	2.034608
22	0.92527541260477					



(a)



(b)

Fig. 2. (a) $1/n$ plots of $\frac{4}{3}B'_n$ (lower) and B_n (upper) which tend to B. (b) Detail of a $1/n$ plot of $B_n \to B \simeq 0.46261$.

here, and below], which tends to B, in the $1/n$ plot of Fig. 2a. However, a more rapid approach to B is achieved by {removing the $[\cdots]$ factor in (15)}

$$B_n = B'_n/[1 - 1/G_1(1 + 1/m)^{\varepsilon_1}] \to B \tag{16}$$

as in the upper graph of Fig. 2a. A $1/n$ plot of B_n in Fig. 2b shows that some $1/n$ dependence remains in this decreasing sequence, and yields the estimate $B \simeq 0.46261$. So from Figs. 2a and 2b, one obtains the approximate formula

$$B_n \sim 0.46261 + 0.0069/n \tag{17}$$

The above analysis suggests it might have been better to write the original sequence in the form

$$c_n + 2 = d_n \sim (A + C_n/G_1^m m^{\varepsilon_1})/G_0^m \tag{18}$$

where $C_n \to C$ as $n \to \infty$, assuming $G_0 = G_1 = 4$, $\varepsilon_1 = -1$, and to calculate the combination (defining C'_n)

$$4^m(A_n - A_{n+1}) \equiv C'_n/m^{\varepsilon_1} = C_n[1 - 1/G_1(1 + 1/m)^{\varepsilon_1}]/m^{\varepsilon_1} \tag{19}$$

directly as in Table II, in which the entries are "clearly" linear in m (as may also be seen from a graph), confirming that $\varepsilon_1 = -1$. For estimation of C, independent of A or B (but with $\varepsilon_1 = -1$), I provide $1/n$ plots of

$$\tfrac{4}{3}C'_n \equiv \tfrac{4}{3} \cdot 4^m(A_n - A_{n+1})/m \text{ (upper), and } C_n \text{ [see (19); lower]} \tag{20}$$

in Fig. 3a, from which $C \simeq 0.1427$. A more rapid approach to the limit $C = 0.142680$ is achieved by C_n, as in Fig. 3b. So from Figs. 3a and 3b one obtains the approximate formula

$$C_n \sim 0.142680 + 0.056/n \tag{21}$$

It is obvious (after a bit of algebra) that we should have

$$B = G_0(1 - 1/G_1)\, C/A \tag{22}$$

where the rhs is 0.462608, providing a better estimate of B.

In summary, the numerical estimates of asymptotic forms for the positions of the principal centres are, with $m = n - 2$,

$$d_n = c_n + 2 \sim A_n/4^m, \qquad \text{with } A_n = 0.925275412602 + mC_n/4^m \tag{23}$$

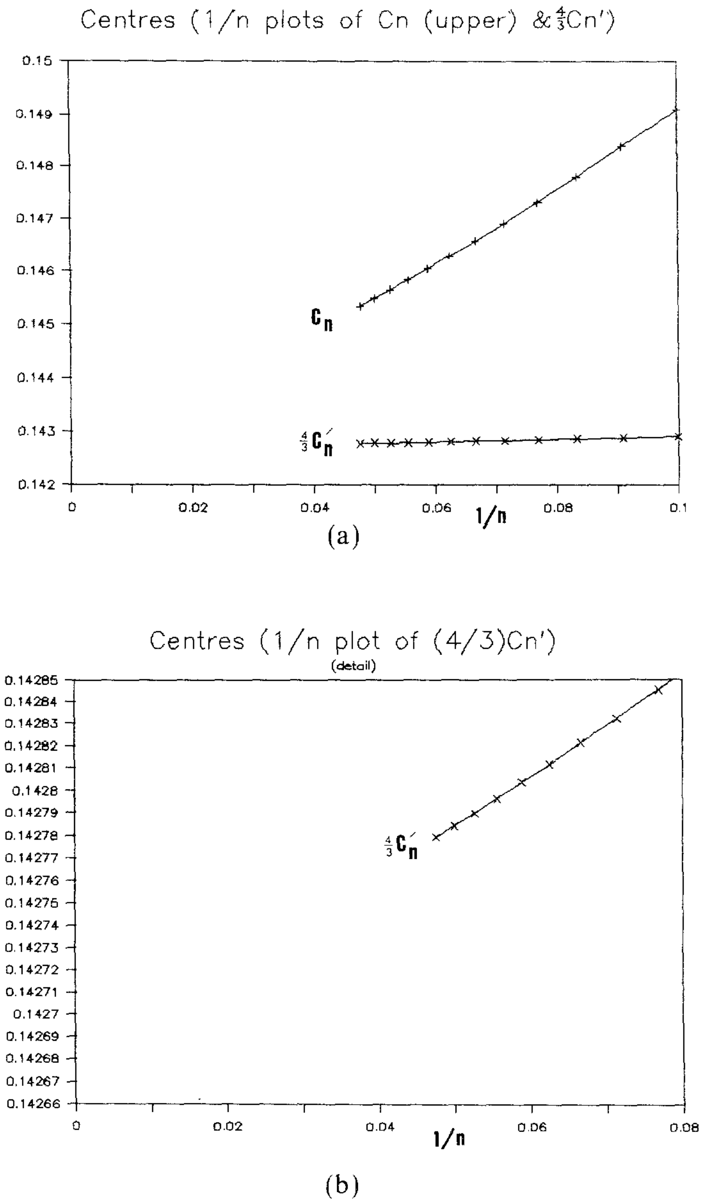


Fig. 3. (a) $1/n$ plots of $\frac{4}{3}C'_n$ (lower) and C_n (upper), which tend to C. (b) Detail of a $1/n$ plot of $\frac{4}{3}C'_n \to C \simeq 0.142680$.

where approximately $C_n \simeq 0.142680 + 0.056/n$, and

$$d_n/d_{n+1} = r_n \sim 4 + mB_n/G_1^m, \qquad \text{with } B_n \simeq 0.462608 + 0.0069/n \quad (24)$$

A better estimate for A is obtained in the next section.

5. ANALYTICAL APPROACH

For very large cycle order n, the cycles involving the centres have the following structure:

$$\begin{array}{ll} z_0 = 0 & x_0 = \frac{1}{2} \\ z_1 = -2 & x_1 = 1 \\ z_2 = 2 & x_2 = 0 \\ z \downarrow 2 & x \uparrow \\ z_{-1} = \sqrt{2} & x_{-1} = \frac{1}{2} - \frac{\sqrt{2}}{4} \\ z_n = 0 & x_n = \frac{1}{2} \end{array} \qquad (25)$$

A typical cycle is shown in the "return map" in Fig. 4.

A partially analytical explanation of the sequence of principal centres can be extracted from Feigenbaum's form of the quadratic map. [I am

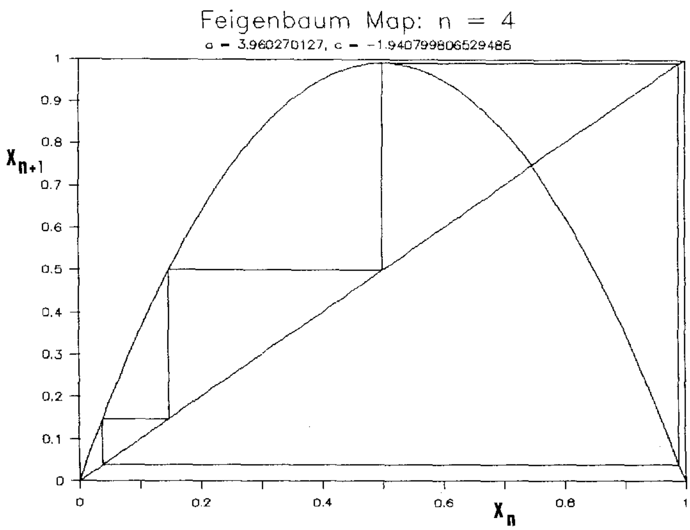


Fig. 4. "Return map" of x_{n+1} versus x_n for the (typical) cycle of order 4 associated with a principal cardioid.

especially grateful to Prof. G. R. Rowlands for suggesting how this could be done, as in (26)–(28) below.] While the cardioids approach the limit point of the Mandelbrot set at -2, the parameter a approaches 4. Consequently, one has, from (3) and (4), for a cycle of order n,

$$a_n = 4(1 - e_n), \qquad d_n = c_n + 2 \sim 6e_n \tag{26}$$

Now starting the cycle (3) at $x_0 = \frac{1}{2}$ and iterating "forwards," we find

$$x_0 = \tfrac{1}{2}, \quad x_1 = 1 - e_n, \quad x_2 = 4e_n(1 - e_n)^2, \quad \ldots$$
$$x_m \sim 4e_n(1 - e_n)^m \, [4(1 - e_n)]^{m-2} \sim e_n 4^{m-1}, \qquad \text{for} \quad m \geqslant 3 \tag{27}$$

since the linear approximation $x_{r+1} \sim ax_r$ can be used when x is small. On the other hand, working "backwards" from $x_n = \frac{1}{2}$, one has from (3) for large n, $x_{-1} \sim (1 - 1/\sqrt{2})/2$. So equating the forward iterate x_{n-1} to the backward iterate x_{-1}, one has to a first approximation for large n

$$x_{n-1} \sim e_n 4^{n-2} \sim (1 - 1/\sqrt{2})/2 \sim x_{-1} \tag{28}$$

whence $d_n \sim 6e_n \sim 6x_{-1}/4^{n-2} \simeq 0.87868/4^{n-2}$, which has the same form as in (23). The approximation can be improved, so after p steps of "backward" iteration to x_{-p} one has

$$x_{n-p} \sim e_n 4^{n-p-1} \sim x_{-p} \tag{29}$$

For a cycle of order n, $x_{-p} = x_{n-p}$. Note that the backward iterates are calculated from (3), so for large index r,

$$x_{r-1} = [1 - (1 - x_r)^{1/2}]/2 \tag{30}$$

starting at $x_n = x_0 = \frac{1}{2}$. On setting $e_n = E_n/4^{n-2}$ and letting $n \to \infty$, one obtains a sequence of estimates for the limiting value of E:

$$E\{p\} = e_n 4^{n-2} \sim 4^{p-1} x_{-p} \to E \tag{31}$$

The corresponding sequence for $A = 6E$ is shown in part in Table III, in which $A\{28\} = 0.925275412602127368$, where the last two digits are doubtful. [35 decimal places were used for x_{-p}, which leaves about 16 for $A\{p\}$.]

The first members of the cycle, x_0, x_1, and x_2, are exact in (27), and a more accurate estimate of the terms for $r \geqslant 3$ can be obtained by using the first-order approximation $x_r \sim e_n 4^{r-1}$ only in the factor $(1 - x_r)$ in the

Table III

p	$A\{p\}$		p	$A\{p\}$		
5	0.9250896	3442760	25	0.9252754	1260212	7202
10	0.9252752	3116357	26	0.9252754	1260212	7328
15	0.9252754	1242494	27	0.9252754	1260212	7360
20	0.9252754	1260195	28	0.9252754	1260212	7368

quadratic map $x_{r+1} = ax_r(1 - x_r)$, as in (3). This yields (the subscript n on e_n is omitted below)

$$\begin{aligned} x_3 &\sim 4^2 e(1-e)^3\,(1-4e) \\ x_4 &\sim 4^3 e(1-e)^4\,(1-4e)(1-4^2 e) \\ &\cdots \\ x_m &\sim 4^{m-1} e(1-e)^m \prod_{r=1}^{m-2} (1-4^r e) \end{aligned} \tag{32}$$

Now set $e_n = E_n/4^{n-1}$ and reverse the order of the product by putting $m = n - p$ and $r = n - q - 2$, so

$$x_{n-p} = E4^{1-p}(1 - E/4^{n-1})^{n-p} \prod_{q=p}^{n-3} (1 - E/4^q) \tag{33}$$

Again let $n \to \infty$ and match up the forward iterate x_{n-p} with the backward iterate x_{-p} to obtain

$$x_{-p} \sim 4^{1-p} E \prod_{q=p}^{\infty} (1 - E/4^q) \tag{34}$$

where for each value of p one gets an estimate $E\{p\}$ for E. The corresponding sequence for $A = 6E$ converges more rapidly than the one from (31).

6. RATIOS INVOLVING TERMS IN ADJACENT CYCLES

Feigenbaum[6] showed that there are two universal numbers δ and α associated with sequences of bifurcations. α determines the scaling of the separation of the terms in the cycles as their order doubles at each bifurcation. There is no quantity completely equivalent to α in the principal cardioid problem. However, there are ratios of term separations which are of some interest. Recall that the centers of the cardioids are located by the values of c for which the origin is a member of the cycle in question. So cycles of order n have terms $z_0 = 0$, $z_1 = c_n, \ldots, z_r, \ldots$, with $z_n = z_0$. The rth term of a cycle of order n with center c_n will be written $z_r(c_n)$, or more simply $z_r(n)$. I will study the ratios of differences between corresponding terms (obtained by the same number of iterations from the starting value z_0) in adjacent cycles (whose orders differ by unity), beginning with the first two cycle members

$$\rho_{12}(n) = [z_2(n) - z_2(n+1)]/[z_1(n) - z_1(n+1)] \tag{35}$$

as a sequence in the cycle order $n = 3, 4, \ldots, \infty$, as in the second column of Table IV. The limit as $n \to \infty$ is $\rho_{12} = -3$. The other columns contain sequences in n for ratios of differences of adjacent (in n) higher-order corresponding terms:

$$\rho_{p,p+1}(n) = [z_{p+1}(n) - z_{p+1}(n+1)]/[z_p(n) - z_p(n+1)] \tag{36}$$

($p = 2, 3, \ldots, 6$). The limits as $n \to \infty$ of these sequences $\rho_{p,p+1}$ are rational numbers (in the bottom row of Table IV) of the form t_{p+1}/t_p where

$$t_p = (-)(2^{2p-1} + 1)/3, \qquad p = 2, 3, 4, \ldots \tag{37}$$

(except for the first column). These numbers would be of little interest, except that they are related to the first derivatives of the iterated maps $R_c^n(0)$ at $c = -2$:

$$t_p = t_p^1 \equiv \partial R_c^p(0)/\partial c \tag{38}$$

[Note that $z_p(n) = R_c^{p-1}(c) \equiv R_c^p(0) = p$th term in a cycle of order n starting at $z_0 = 0$, and as $n \to \infty$, $c_n \to -2$.] These derivatives appear in a Taylor expansion of $R_c^n(0)$ in (5), for a cycle of finite order n [replacing p in (38)] with parameter c_n at a center, about $c = -2$:

$$\begin{aligned} R_c^n(0) &= R_c^n(0)\Big|_{c=-2} + (c_n+2)\frac{\partial R_c^n(0)}{\partial c}\Big|_{c=-2} + \frac{1}{2}(c_n+2)^2\frac{\partial^2 R_c^n(0)}{\partial c^2}\Big|_{c=-2} \cdots \\ &= 2 + d_n \cdot t_n^1 + \tfrac{1}{2} d_n^2 \cdot t_n^2 + \cdots \end{aligned} \tag{39}$$

From the first two terms of this series (in d_n), we get an estimate [the series $= 0$ at c_n by (5)]

$$c_n + 2 = d_n \sim -2/t_n^1 \sim 12/4^n = (3/4)/4^{n-2} \sim A/4^{n-2} \tag{40}$$

Table IV. Sequences in n for $\rho_{p,p+1}(n)$: Ratios of Differences of Adjacent (in n) Higher-Order Terms, As in (36)

n	ρ_{12}	ρ_{23}	ρ_{34}	ρ_{45}	ρ_{56}	ρ_{67}
3	−2.695678	—	—	—	—	—
4	−2.926224	3.440652	—			
5	−2.981800	3.610259	3.709825	—		
6	−2.995472	3.652592	3.858974	3.784002	—	
7	−2.998870	3.663151	3.896548	3.928187	3.802972	—
8	−2.999718	3.665788	3.905955	3.964583	3.945970	3.807739
9	−2.999929	3.666447	3.908307	3.973702	3.982082	3.950445
10	−2.999982	3.666612	3.908895	3.975984	3.991133	3.986489
11	−2.999996	3.666653	3.909042	3.976554	3.993397	3.995523
12	−2.999999	3.666663	3.909079	3.976697	3.993963	3.997782
13	−3.000000	3.666666	3.909088	3.976732	3.994105	3.998348
14	−3.000000	3.666666	3.909090	3.976741	3.994140	3.998489
—	—	—	—	—	—	—
∞	−3	11/3	43/11	171/43	683/171	2731/683

where (very roughly) $A \sim 3/4$. Although this expression has the same form as (23), it is not correct, for we cannot justify dropping the higher-order terms in the series (39). The third term involves the second derivative:

$$t_n^2 = \left. \frac{\partial^2 R_c^n(0)}{\partial c^2} \right|_{c=-2} \overset{?}{=} [2 \cdot 16^{n-1} + 6(n-1)\, 4^{n-1} - 2]/27 \qquad (41)$$

where the formula (proved below) checks for $n = 3$–7 with the numerical values obtained from the exact polynomials, which are constructed by algebraic iteration. Now keeping only the leading order terms in t_n^1 and t_n^2, and writing $d_n = A/4^{n-2}$, we find that the expansion (39) for $R_c^n(0)$ involves only A:

$$R_c^n(0) = 2 - \tfrac{8}{3}A + \tfrac{16}{27}A^2 + \cdots = 0 \qquad (42)$$

which yields (as a quadratic in A)

$$A \sim \tfrac{1}{4}(9 - \sqrt{27}) = 0.9509619 \cdots \qquad (43)$$

as an improved estimate for the amplitude. The second term on the rhs of (41) has the form $n4^n$, and is responsible for the exponent $\varepsilon_1 = -1$ in (12) and in Section 4. At this point one may ask whether it is possible to extend the series to higher derivatives. It is: see below!

7. EXACT SOLUTION OF THE PRINCIPAL CARDIOID PROBLEM

We now derive the formulas (37) and (41) for t_n^1 and t_n^2, and obtain the entire series (39) in d_n for asymptotically large n. Writing R^n for $R_c^n(0)$, the basic recurrence relation becomes

$$R^{n+1} = (R^n)^2 + c \qquad (44)$$

so at $c = -2$, $R^n = 2$ for $n > 1$. Denoting differentiation with respect to c by $D \equiv \partial/\partial c$, one has

$$DR^{n+1} = 2R^n \cdot DR^n + 1 \qquad (45)$$

On setting $t_n^1 = DR^n$ at $c = -2$, (45) yields a recurrence relation

$$t_{n+1}^1 - 4t_n^1 = 1 \qquad (46)$$

starting at $t_2^1 = 2c + 1|_{c=-2} = -3$, for the first derivative. The solution is, as in (37) and (38),

$$t_n^1 = DR_n|_{c=-2} = (-)\,\tfrac{1}{3}(\tfrac{1}{2}4^n + 1), \qquad n = 2, 3, 4, \ldots \qquad (47)$$

Next differentiate (45) to get

$$D^2R^{n+1} = 2(DR^n)^2 + 2R^n \cdot D^2R^n \tag{48}$$

which becomes a recurrence relation for the second derivative $t_n^2 = D^2R^n$ at $c = -2$:

$$t_{n+1}^2 - 4t_n^2 = 2[\tfrac{1}{3}(\tfrac{1}{2}4^n + 1)]^2, \qquad \text{with} \quad t_2^2 = 2 \tag{49}$$

The solution is, as in (41)

$$t_n^2 = \tfrac{1}{27}[\tfrac{1}{8} \cdot 4^{2n} + \tfrac{3}{2} \cdot (n-1)\, 4^n - 2], \qquad n = 2, 3, 4... \tag{50}$$

The rhs of the difference equation (49) contains a term 4^n which is also a solution of the homogeneous difference equation (the complementary function), which explains why the solution contains a term like $n4^n$, and hence why $\varepsilon_1 = -1$ in Section 4.

It is important to notice that only the leading order terms contribute to the expansion of $R_c^n(0)$ in powers of $A = 4^{n-2}d_n$ in (39), as in (42). For a derivative of order r the leading term involves 4^{nr}. Moreover, for $r > 1$ these terms arise from particular solutions to the difference equations. Therefore to obtain the expansion in powers of A it is sufficient to retain only the largest terms on the rhs of the difference equations. For a particular order of derivative, the rhs are constructed from lower-order derivatives. Consequently, the third and higher derivatives at $c = -2$ only have to be calculated to leading order. This simplifies matters considerably!

Let us list the relations which express the rth derivatives of R_c^{n+1} in terms of derivatives of R_c^n of equal or lower orders for $n > r$ [$D \equiv \partial/\partial c$, and the sub and superscripts c and n are dropped from R_c^n on the rhs]:

$$\begin{aligned}
D^3R_c^{n+1} &= 2R \cdot D^3R + 6DR \cdot D^2R \\
D^4R_c^{n+1} &= 2R \cdot D^4R + 8DR \cdot D^3R + 6(D^2R)^2 \\
D^5R_c^{n+1} &= 2R \cdot D^5R + 10DR \cdot D^4R + 20D^2R \cdot D^3R \\
D^6R_c^{n+1} &= 2R \cdot D^6R + 12DR \cdot D^5R + 30D^2R \cdot D^4R + 20(D^3R)^2 \\
D^7R_c^{n+1} &= 2R \cdot D^7R + 14DR \cdot D^6R + 42D^2R \cdot D^5R + 70D^3R \cdot D^4R \\
D^8R_c^{n+1} &= 2R{\cdot}D^8R + 16DR{\cdot}D^7R + 56D^2R{\cdot}D^6R + 112D^3R{\cdot}D^5R + 70(D^4R)^2
\end{aligned} \tag{51}$$

[See also (44), (45), and (48).] Evaluating all the relations in (51) at $c = -2$, so $R = 2$ ($n > r > 2$ here), and setting

$$t_n^r = D^rR^n \qquad \text{at} \quad c = -2, \quad (n = \text{cycle order}) \tag{52}$$

one obtains a sequence of linear first-order difference equations of the form

$$t^r_{n+1} - 4t^r_n = \text{RHS}, \qquad \text{where} \quad \text{RHS} \sim (\text{rational number}) \cdot 4^{nr} \tag{53}$$

In each case for $n > r > 2$ only the particular solution is required, for large cycle order n:

$$\begin{aligned}
t^1_n &\sim -4^n/2 \cdot 3 \\
t^2_n &\sim +4^{2n}/2^2 \cdot 3^2 \cdot 2 \cdot 3 \\
t^3_n &\sim -4^{3n}/2^3 \cdot 3^3 \cdot 3 \cdot 4 \cdot 5 \\
t^4_n &\sim +4^{4n}/2^4 \cdot 3^4 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \\
t^5_n &\sim -4^{5n}/2^5 \cdot 3^5 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \\
t^6_n &\sim +4^{6n}/2^6 \cdot 3^6 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \\
t^7_n &\sim -4^{7n}/2^7 \cdot 3^7 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \\
t^8_n &\sim +4^{8n}/2^8 \cdot 3^8 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15
\end{aligned} \tag{54}$$

There is obviously a simple rule for the denominators (checked up to t^{10}_n):

$$t^r_n \sim (-4^n)^r \cdot (r-1)!/2^r \cdot 3^r \cdot (2r-1)!, \qquad r = 1, 2, 3, 4,... \tag{55}$$

So the Taylor expansion for $R^n_c(0)$ about $c = -2$, as in (39), becomes, to leading order in each term,

$$\begin{aligned}
R^n_c(0) &= 2 + \sum_{r=1}^{\infty} \frac{d^r_n}{r!} \frac{\partial^r R^n}{\partial c^r}\bigg|_{c=-2} \\
&\simeq 2 + \sum_{r=1}^{\infty} \frac{d^r_n t^r_n}{r!} \\
&= 2 + \sum_{r=1}^{\infty} \frac{(-4^n d_n/6)^r \, (r-1)!}{r! \, (2r-1)!} \\
&= 2 \cos[(8A/3)^{1/2}] \ \cdots \ !
\end{aligned} \tag{56}$$

The relevant zero is at $(8A/3)^{1/2} = \pi/2$, so

$$A = 3\pi^2/32 = 0.9252754126021273705 15 \cdots \tag{57}$$

By keeping the next-order terms in the difference equations and their solutions, once could evaluate the constants in the asymptotic expressions (23) and (24) for the centers of the principal cardioids.

8. OTHER ITERATIVE MAPS

Similar sequences appear for other iterative maps of the form $x_{n+1} = F(x_n)$. Rowlands[7] (quoted with slight changes in notation) has shown that, "for the map:

$$F(x;a) = \begin{cases} ax, & 0 \leqslant x \leqslant \frac{1}{2} \\ a(1-x), & \frac{1}{2} \leqslant x \leqslant 1 \end{cases} \tag{58}$$

[for which the limiting value of a is $a_\infty = 2$] setting a $[= a_n = a_\infty(1 - e_n)] = 2(1 - e_n)$ and $e_n = E_n/2^n$, one gets an exact equation

$$E_n(1 - E_n/2^n)^{n-1} = 1 \tag{59}$$

which ... gives $E_n \to E = 1$. However, this map is not of the Feigenbaum class. In all cases

$$e_n \sim E_n/[F'(0)]^n, \qquad \text{where} \quad F'(0) = F'(x=0; a=a_\infty), {}' = \partial/\partial x \tag{60}$$

which is not too surprising since (x) spends a lot of iterates in the linear regime of the map, where $x_{r+1} \simeq F'(0) \cdot x_r$."

The terms in the required sequence are exactly

$$x_0 = 1/2, \qquad x_1 = 1 - e_n, \qquad \ldots, \qquad x_r = e_n[2(1-e_n)]^{r-1}, \qquad 2 \leqslant r \leqslant n \tag{61}$$

Consequently, the cycle closes at order n if $x_n = x_0 = \frac{1}{2}$, so

$$e_n[2(1-e_n)]^{n-1} = 1/2 \tag{62}$$

which determines e_n, and is equivalent to (59). For increasing n it is easy to obtain a sequence of values of e_n (or E_n) by solving (62) (or (59)) numerically. Then $E_n \to 1$ and $a_n \to 2$.

9. CONCLUDING REMARKS

The principal cardioid problem for the Mandelbrot set discussed in this paper shows the power of numerical methods, such as the ratio method, in determining the asymptotic behavior of the terms in a sequence. A scaling reduction by a factor of 4 has been established for the distances of the centers of the relevant cycles from the limit point at -2. One would expect that a similar scaling factor will apply to the longitudinal and transverse dimensions of the cardioids in the complex plane.

We have examined the sequence of centers of cardioids which are closest to the limit point. There will be other, similar sequences of (centers of) cardioids corresponding to cycles of increasing order along the real axis. For example, there will be a sequence of cardioids which are "second closest" to the limit point, then another sequence which is "third closest," and so on. Each of these sequences may be indexed by its "level" of proximity to the limit point. These higher-level sequences will begin only when the initial cycle order is sufficiently high. For example, the cycle of order five yields three cardioids on the real axis, one of which belongs to the "principal" sequence. The other two cardioids may well signal the beginning of the next two "levels" of sequences. It is tempting to suggest that the scaling factor from the limit point is still 4, and that the "amplitudes" of these sequences will be given exactly by the higher-order zeros of the cosine function in (56)! If this turns out to be the case, then we will be able to classify *all* possible sequences of cardioids by their "levels." Calculations for cycles at level 2 confirm the above conjecture for the sequence of cardioids "second closest" to the limit point. Further study of such sequences is in progress.

NOTE ADDED IN PROOF

The difference equations in (51) and their solutions in (54) can be verified by induction. It is easy to see that the "levels" l correspond to all possible "inverse orbits" from -2, arranged in such an order that $z_0 \simeq (-)^l \pi D(2l-1)/4^n$, where $D \equiv \partial R_c^n/\partial z$. At points on the boundary of the Mandelbrot set D has an absolute value of unity[3] so $D = \exp(i\theta)$, say. Then one can derive a parametric formula for all the cardioids: $c_n - c_n(\text{centre}) \simeq \frac{1}{2}e^{i\theta}(1 - \frac{1}{2}e^{i\theta})\, S/16^{(n-1)}$, with $S = \frac{3}{8}[\pi(2l-1)]^2$.

ACKNOWLEDGMENT

I would like to thank Prof. G. R. Rowlands for his contributions to this work, and for several illuminating discussions and letters, which have helped convert intuition into understanding.

REFERENCES

1. C. Domb, *Adv. Phys.* **9**:149 (1960); **19**:339 (1970).
2. C. Domb and M. F. Sykes, *J. Math. Phys.* **2**:63 (1961).
3. B. B. Mandelbrot, *Ann. N.Y. Acad. Sci.* **357**:249 (1980).
4. H.-O. Peitgen and P. H. Richter, *The Beauty of Fractals* (Springer-Verlag, New York, 1986).
5. R. M. May, *Nature* **261**:459 (1976).
6. M. J. Feigenbaum, *J. Stat. Phys.* **19**:25 (1978); **21**:669 (1979).
7. G. R. Rowlands, private communications (1989).